8.2 - Homogeneous Linear Systems

Consider the system $\begin{aligned}\frac{dx}{dt} &= 2x + 2y \\ \frac{dy}{dt} &= x + 3y\end{aligned}$ which, in $\mathbf{X}' = \mathbf{A}\mathbf{X}$ form is

$\mathbf{X}' = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \mathbf{X}.$

Example: Find the general solution of the given system.

$$\frac{dx}{dt} = 2x + 2y$$
$$\frac{dy}{dt} = x + 3y$$

Example: Find the general solution of the given system.

$$\frac{dx}{dt} = 2x - 7y$$
$$\frac{dy}{dt} = 5x + 10y + 4z$$
$$\frac{dz}{dt} = 5y + 2z$$

Theorem: Let $\lambda = \alpha + i\beta$ be an eigenvalue of $\mathbf{A}$ and let $\mathbf{B}_1 = \text{Re}(\mathbf{K})$, $\mathbf{B}_2 = \text{Im}(\mathbf{K})$. Then

$$\mathbf{X}_1 = e^{\alpha t}(\mathbf{B}_1 \cos \beta t - \mathbf{B}_2 \sin \beta t)$$
$$\mathbf{X}_2 = e^{\alpha t}(\mathbf{B}_2 \cos \beta t + \mathbf{B}_1 \sin \beta t)$$

Example: Find the general solution of the given system.

$$\frac{dx}{dt} = 4x + 5y$$
$$\frac{dy}{dt} = -2x + 6y$$

Repeated eigenvalues:

Example: Find the general solution of the given system.

$$\mathbf{X}' = \begin{pmatrix} 12 & -9 \\ 4 & 0 \end{pmatrix} \mathbf{X}$$

Example: Find the general solution of the given system.

$$\frac{dx}{dt} = 3x + 2y + 4z$$
$$\frac{dy}{dt} = 2x + 2z$$
$$\frac{dz}{dt} = 4x + 2y + 3z$$

As shown in the diagram, two large connected mixing tanks A and B initially contain 100 gallons of brine. Liquid is pumped in and out of the tanks as indicated in the figure; the mixture pumped between and out of a tank is assumed to be well-stirred. Construct a mathematical model in the form of a linear system of first-order differential equations for the number of pounds $x_1(t)$ and $x_2(t)$ of salt in tanks A and B, respectively, at time t. Write the system in matrix form.

AI response to where this model is used: This type of coupled mixing tank model shows up in several real engineering and scientific contexts:

Environmental & Water Treatment The most direct application is **wastewater treatment plants**, where tanks are linked in series or loops to progressively dilute or react contaminants. The exact flow-in/flow-out balance you see here models how pollutant concentrations evolve through each stage.

Pharmacokinetics (Drug Modeling) This is probably the *most important* real use. The human body is modeled as a series of "compartments" — blood plasma, tissue, organs — and a drug moves between them at known rates. The math is identical to these tank equations. Pharmacologists use this to predict:

- How long a drug stays in your system
- When blood concentration peaks
- Proper dosing intervals
- Chemical Reactors

In industrial chemistry, **continuous stirred-tank reactors (CSTRs)** are literally this setup. Reactants flow in, products flow out, and intermediate products recirculate between vessels. Petroleum refining and pharmaceutical manufacturing use multi-tank CSTR networks constantly.

Groundwater Contamination Aquifers are modeled as interconnected "cells" exchanging water at known rates. When a contaminant spills, hydrologists use these equations to predict how the plume spreads between underground zones over time.

HVAC and Atmospheric Modeling The atmosphere itself is divided into compartments (troposphere, boundary layer, urban airshed) with pollutants like CO_2 or smog transferring between them — same mathematical structure.

The reason this model is so powerful is that **any system where a substance moves between well-mixed reservoirs at proportional rates** reduces to the same form of coupled linear ODEs. The tanks are just the most intuitive way to introduce the concept.